Juggling: Theory and Practice

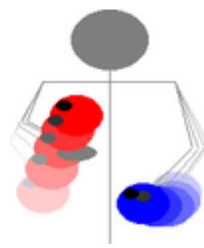
by Colin Wright

Juggling has fascinated people for centuries. Seemingly oblivious to gravity, the skilled practitioner will keep several objects in the air at one time, and weave complex patterns that seem to defy analysis. As the first known depiction of jugglers dates back nearly 4000 years, it's hard to imagine there's anything new to learn.

But a lesson we've learned from Martin Gardner is that there's always something new to learn, always something new to discover. So let's start with a quick review of classical juggling, and then see what new things we found, partly by accident, partly by hard work, and mostly because with mathematics we can see things that are otherwise hidden. We start by describing briefly the classic juggling patterns.

The Standard Pattern

The most common misconception is that when we juggle, the balls go round in a (highly elongated) circle. Juggling the balls in a cycle like this requires that every time a ball is thrown it must be handled by each hand (and therefore at most twice for all but the exceptionally gifted). In particular, the hands do different things. One hand catches the ball and shunts it over, the other hand receives the ball and then launches it into the air.



Exercise: ignoring air resistance, what are the paths of the balls? Warning: It's not a parabola.

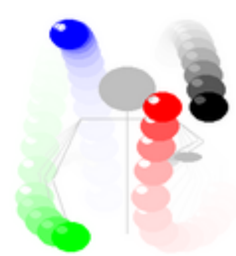
Anyone who tries this with two balls will launch with their dominant hand, showing clearly that when juggling the **throw** is critically important, not the catch. If the throw is perfect, the catch will take care of itself.

So what happens if we ask that each hand does the same thing, and each ball does the same thing (as each other ball, not the same as the hands. That would be silly).

Firstly, each ball must be thrown in turn. If not, then one ball must have overtaken another, so their throws aren't the same. If we're juggling ***n*** balls, and each ball is thrown in turn, it then becomes clear that there are two distinct cases: either the number of balls is divisible by the number of hands, or it isn't.

Concentrating on the case of the juggler with exactly two hands, that give us two distinct cases: an even number of balls, and an odd number of balls.

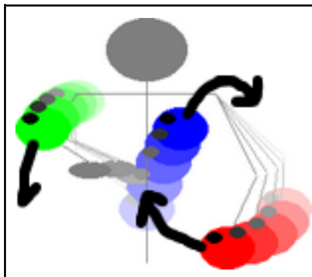
If we're juggling an even number of balls, and the balls are all thrown in turn, each ball will have to stay in the same hand. The pattern then becomes half the balls in each hand and is referred to as the "Fountain" pattern. (Pictured here at right.) Although often regarded by non-jugglers as somehow being akin to cheating, the analysis above shows that the non-changing of hands is required by the condition that all throws be the same.



Four ball fountain

Equally, if we're juggling an odd number of balls, each ball must be thrown by alternate hands. This leads to a "Lazy-Eight" pattern known (in English) as the "Cascade". Again, with an odd number of balls, the condition that every throw be the same requires us to have each ball changing hands.

These we call the "Standard Pattern" for a given number of balls. Other conditions are often imposed for basic patterns, most commonly that throws (and hence catches) occur in a metronomic rhythm, and throws are made inside shoulder width, and catches are made outside shoulder width.



Three ball cascade

Juggling in theory

So some simple analysis can tell us things about juggling patterns that we might otherwise not realise. One of the earliest published examples of this came from Claude Shannon, the father of Information Theory. The "Shannon Juggling Theorem" says that when juggling the standard pattern we have ***b(d+e) = h(d+f)*** where ***b*** is the number of balls, ***d*** is the dwell time (the time a ball spends in the hand), ***e*** is empty time of a hand, ***h*** is the number of hands, and ***f*** is the flight time of a throw. [1]

But what about non-standard patterns? There are literally infinitely many possible variations on a theme. The throws and catches can be made in many different places, the timing can be varied, balls can be carried through and around other balls, multiple balls can be thrown and caught together, and so on.

The possible variations are so great both in style and detail that it is unsurprising that, despite thousands of years of history, until the mid-1980s there was no notation for juggling patterns. Even then the vast array of possibilities seemed to make the task of devising a notation impossible.

Simplifying juggling

To make progress we simplify the domain of discourse. Specifically, we assume from here on that throws happen to a metronomic beat, and that the throws and catches happen just as for the standard pattern. Further, we assume that we only throw one ball at a time, and we only catch one ball at a time.

Most people who start juggling want to learn four, then five, and so on. Here is the secret to learning to juggle five - don't practice five! Instead, practice each required skill separately. For each skill - hand speed, throw angle accuracy, throw height accuracy, rate of throw, *etc.* - find a simpler trick that requires that skill and practise that trick. After finding and mastering a trick for each separate skill, putting them together becomes achievable in a much shorter time than trying to acquire all the skills ~~symultaneously~~ ~~simultaniously~~ ~~similtayneusly~~ at the same time. And it's more fun.

We're now left with very few options for finding variations in our juggling. Specifically, when we throw a ball, the time it spends in the air is quantised, because it has to come down at one of the prescribed times for catches. Further, once we know *when* it comes down, that controls how *high* it goes, and *where* it comes down. So we describe each throw by a single number - the time it spends in the air. However, since we don't know what proportion of time the hands spend full (or empty), it makes our task easier to think not of the catch, *but of the next throw.* Now we can see that because throws are separated by a whole number of beats, each ball spends a whole number of beats in its journey from one throw to the next. Each throw can be described entirely by this single number.

The magical thing about this number is that when we're juggling three balls in the standard pattern, each ball is thrown every third time, so the number to describe the throw is a 3. And there's nothing special about 3. Whenever we juggle the standard pattern for ***n*** balls, each ball is thrown every n^{th} throw.

Back in the mid 1980s it was realised [2] that some of the well-known juggling tricks could be described completely just by the appropriate string of numbers to describe the throws. Obviously when juggling the standard three ball pattern we can write ...3333... and for 8 balls we can write ...8888... and so on, but there is a well-known trick with four clubs. Normally juggled with double spins, throw one club high with a triple spin, and the next club low with a single, each club changing hands. Each club drops into the slot vacated by the other, and the pattern then continues as if nothing happened. Much less impressive when done with balls, it is a useful exercise to practise the exact height required for five ball juggling. The high throw will next be thrown five beats later, so is described as a five. The low throw is a three, so we can describe a single instance of this trick as ...444_53_444...

Another well-known four ball trick is make two consecutive throws as if juggling five, pause, and then restart. This can be described as ...444_552_444... (Exercise: why is a momentary hold described as a 2?)

Another variation is to throw all four balls as if juggling five. Of course, after the first four throws we've run out of balls, but if we wait for a beat all the balls come down in order and we can restart our four ball pattern. We write this as ...444_55550_444... It's no surprise that for that moment when we don't have a ball we describe it as a 0, although we shall shortly see that this raises some interesting questions.

Putting it all together ...

Collecting these different tricks and writing them one above the other, putting at the top the uninterrupted four ball fountain, we end up with this:

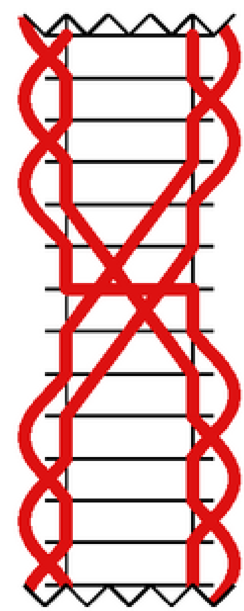
... 4 4 4 **4** 4 4 4 ...
... 4 4 4 **5 3** 4 4 4 ...
... 4 4 4 **5 5 2** 4 4 4 ...

... 4 4 4 **5 5 5 5 0** 4 4 4 ...

The pattern was almost impossible to see when we first wrote these down, but leaving the gap makes it unmistakable. The pattern ... 444 **5551** 444 ... is clearly missing, and based on the sequence, clearly should be a juggling trick.

But it was a trick we didn't know.

From a four ball fountain throw three balls as if juggling a five ball cascade. Now you have one ball left - DON'T THROW IT! Zip that ball across into the otherwise empty hand. Now instead of waiting for a beat, you can carry on immediately.



Space-time diagram for 5551

Do this constantly, and suddenly it feels a lot like five balls. Three out of every four throws is a 5-ball throw, and the pattern is there, in the air, with just a flicker for a missing ball every fourth beat. Superb practice for 5, and enormously easier as it's only four.

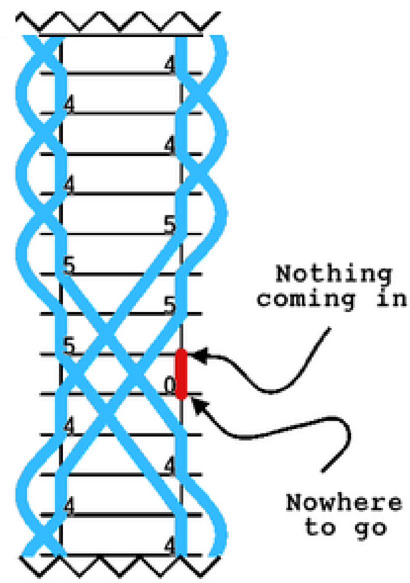
An entirely new juggling trick, discovered through mathematics.

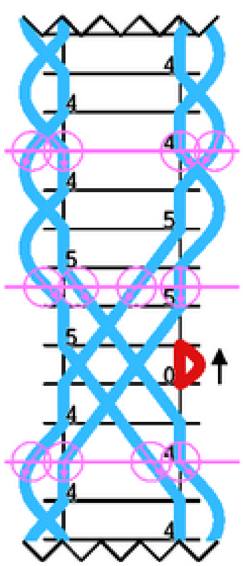
A mystery emerges ...

We've shown that some juggling tricks can be described by sequences of numbers, and that by following patterns we can find previously unknown tricks. Not all sequences make valid juggling tricks, but space does not permit investigation of that particular aspect.

There is another question that emerges, however, when we look at the physical reality. We do, after all, have to hold the ball between catching and throwing. In our previous Space-Time diagram we've made the simple assumption that the hands are full for exactly half the time, and we can see that the throws that come back to the same hand are four beats from throw to throw, three beats in the air, giving a hold time of one. The high throws that change hands are fives, and they spend four beats in the air. The zip across is no time in the air, one beat in the hand, and therefore its "Cycle Time" - the time to the next throw - is 1. All this is just as we might expect.

But what about the 0 in 55550? Every other number is the time from throw to throw, and the time in the air is one less. If we follow that pattern, the 0 should give an air-time of -1. We have predicted the time-travel of a juggling ball. How can that possibly work??

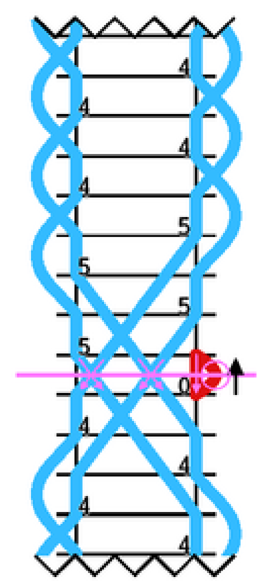




If the hands are full for half the time we end up with a ball in the right hand that has come from nowhere, and has nowhere to go. Clearly we should have the ball go back in time to become itself, just as required.

If we draw a horizontal line on our diagram it's a single instant of time, dividing past above from future below. In a sense it's a photograph, freezing the action and seeing where things are. The diagram here at left has several photographs, each showing where all four balls are.

In each case there's a ball in the hand and balls in the air, always exactly four of them. Which is right and reasonable, as we are juggling four balls. By the conservation law of juggling equipment we should always have four balls.



But look at the photograph in the diagram on our right. Here we have four balls in the air between the hands, and another ball in the right hand. Clearly there's something strange happening. But wait! There's more! There's also a ball going backwards in time. That must count as a negative ball, to bring our count back to the required four.

It's an anti-ball!

We can think of the "catch" (where the ball comes from the future) as the mutual creation of a ball/anti-ball pair, and the throw back into the past as the mutual annihilation. Thus we have confirmed the view in modern physics that an anti-particle can be thought of as a particle going backwards in time: a positron is an electron going backwards in time, an anti-proton is a proton going backwards in time, *etc.* More, since a photon is its own anti-particle it doesn't know whether it's coming or going, but since it travels at the speed of light, Einstein tells us time is stopped.

But $E=mc^2$, so where does the energy come from to create a ball/anti-ball pair? Just as there's a quantum uncertainty principle between position and momentum, there's also a quantum uncertainty principle between energy and time. We know exactly when the throws and catches are happening, so we have a very small uncertainty in time and we can borrow from the quantum uncertainty in energy to create a virtual ball/anti-ball pair.

In truth, the anti-ball can be thought of as subtracting a ball from where we expect one, leaving us with an empty hand when our assumptions would normally require a ball.

And in conclusion ...

It doesn't end there. Now there are notations for hand movements, timing variations, patterns involving more than one juggler. We have arithmetic methods for determining whether a given sequence can be juggled, and algorithms for producing all possible juggling sequences with any number of balls. Work continues to make these newer notations simpler, cleaner, and more useful.

But the real bonus is that this material is being used as a vehicle to bring the excitement and enthusiasm of recreational mathematics to thousands each year, year on year.

The juggling is fun, but the maths, as one student said to me, is "funner".

Footnotes / References

1. A draft paper for Scientific American is included in "Claude Elwood Shannon Collected Papers," edited by N.J.A. Sloane and A. D. Wyner, New York, IEEE Press, 1993, pages 850-864).
2. http://www.solipsys.co.uk/new/Juggling.html
3. http://www.solipsys.co.uk/new/ColinWright.html
4. http://en.wikipedia.org/wiki/Siteswap
5. http://www.cecm.sfu.ca/organics/papers/buhler/paper/html/paper.html

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The Author

Colin Wright took his B.Sc. at Monash University, Australia, and his Ph.D. at Cambridge University, UK, both in Pure Mathematics. These days he is Director of Research at a company which makes maritime surveillance equipment, still finding time to give presentations all over the world on "Juggling - Theory and Practice."