How High the Moon

A paper for G4G9

by Colin Wright

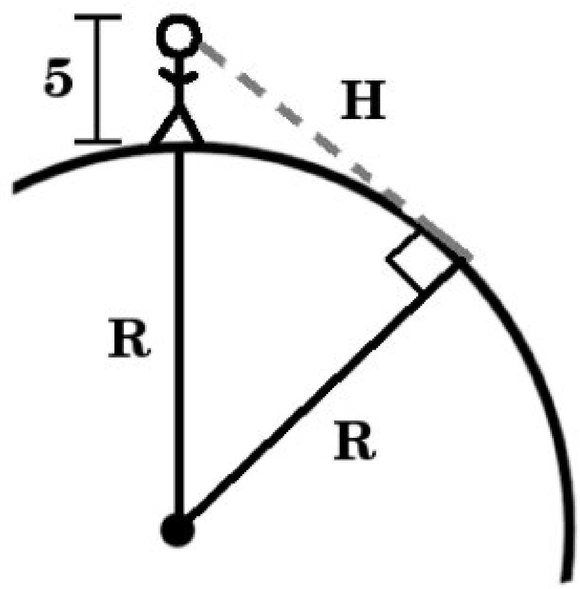
We're going to compute the distance to the Moon using a few well-known facts, a few simple observations, a pendulum, and a stopwatch. Pretty much everything here was known to Isaac Newton in the late 1600's, and it's even been suggested that he performed pretty much exactly these calculations.

Maybe, maybe not. Let's just see what we can do with some really elementary reasoning.

We'll warm up with a well-known (in some circles!) question: How far is the horizon?

> Some time ago while drifting off to sleep it seemed like a bunch of stuff I knew all tied up together into a neat bundle.
>
> Then I woke up.
>
> Surprisingly, it all still worked! Here are the results.

We'll pretend things are simple. We'll pretend the Earth is a sphere, and suppose we're at the top of a tall mountain, say, 5 metres high. (Yes - I know that's not very tall really, but bear with me ...)



We can create a right-angled triangle with one corner at the centre of the Earth, one corner at our position, and one where our line-of-sight tangents the Earth's surface.

Our good friend Pythagoras now steps up and says that $R^2+H^2=(R+5)^2$, which can be expanded and simplified and we get

$$H^2 \approx 10 \times R.$$

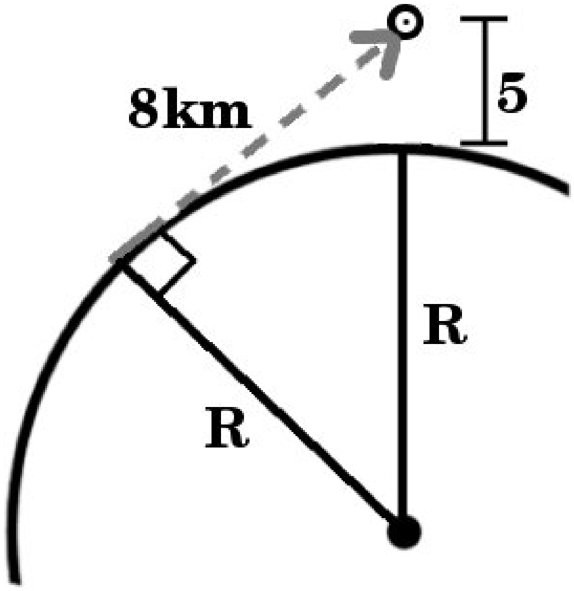
(As in this case, I'll keep the equations in the main text fairly simple throughout and expand on them in the boxes on the right side of the page. Ignore them if you just want the main ideas, or if you want the challenge of working out the details yourself.)

> $R^2+H^2=(R+5)^2$
> $R^2+H^2=R^2+2x5xR+5^2$
> $R^2+H^2=R^2+10xR+25$
> so
> $H^2=10xR+25$
> Then we ignore the 25.

Now, the original definition of the metre was "One ten-millionth of the distance from the North Pole to the Equator through Paris," which means the circumference of the Earth is 40 million metres, so the radius is roughly 6.4 million metres. Substituting this we get

$$H^2 \approx 64 \times 10^6, \text{ and so } H \approx \sqrt{64 \times 10^6} = 8000m$$

So from a height of 5 metres, the distance to the horizon is about 8000 metres, or 8km.



Now let's turn it around. Suppose we're at sea level and 8000m from the top of a 5m high mountain. Suppose further we fire a projectile line-of-sight at the peak, ignore air resistance, and it gets there in 1 second (unlikely, I know). In one second it will fall about 5m, because acceleration due to gravity is about $10 m/s^2$, so by the time it gets there, it will still be at sea-level. In other words, it will be grazing the Earth's (perfectly spherical) surface.

It's in orbit.

So we've just shown that subject to all our approximations, orbital velocity at grazing altitude is 8 km/s. Quite astonishing how our good friend Pythagoras is, in some sense, "Rocket Science."

So now let's be a little more general. Instead of being exactly 5m high, let's pick an acceleration ***a*** and an amount of time ***t*** and suppose we are $at^2/2$ high.

That seems arbitrary, I know, but distance fallen in time ***t*** under acceleration ***a*** is given by ***d=at^2/2*** so we're at a height such that something will fall that distance in time ***t.***

Our Pythagorean triangle equation now becomes

$$H^2+R^2=(R+at^2/2)^2$$

$$H^2+R^2=(R+at^2/2)^2$$
$$H^2+R^2=R^2+2\times R\times at^2/2+(at^2/2)^2)$$

Ignoring the last term (because it's small) and cancelling the R^2 term, we get:

$$H^2=2\times R\times at^2/2$$
$$H^2=R\times a\times t^2$$
$$(H/t)^2=R\times a$$
$$v^2=R\times a$$

We simplify, divide through by t^2, throw away the irrelevant small part, and then remember that distance over time is velocity. That means we get

$$v^2=aR$$

or equivalently,

$$a=v^2/R$$

Suddenly we have the formula for acceleration in a circle.

Which is nice.

What does this have to do with the Moon?

You may ask why something moving in a circle is accelerating? Well, its speed may not be changing, but its direction is. If left alone its direction wouldn't change, so something must be pushing on it, changing its direction. Newton tells us that Force is Mass times Acceleration, so if there's a force, there must be acceleration.

I've also been somewhat cavalier about ignoring small quantities and so forth. In truth there are some details about limits and such like, and that's where the serious calculus should be done.

If we suppose the Moon to be moving in a circular orbit, and supposing the radius of that orbit to be ***M,*** we can now say that its acceleration in orbit is v^2/M. So if only we knew how far away it was, and its velocity, we would know its acceleration.

But if we know its distance then we do know its velocity, because we know it takes 29.53 days from full moon to full moon. Correcting for sidereal time, that means it takes 27.32 days to make a complete circuit of the Earth. Call that time ***P.*** Therefore the Moon's velocity in orbit is

$$(2\pi M)/P$$

When the Moon goes around the Earth, the Earth is also going around the Sun. In the 365.25 days it takes for a year the Moon goes around the Earth - apparently - some 365.25/29.53 times. However ...

The orbit of the Earth itself adds another complete rotation. From the point of view of the stars the Moon hasn't gone around 12.37 times, it's gone around 13.37 times, and that means that each orbit, from the point of view of the stars, takes 365.25/13.37 days, or 27.32 days.

Thus the <u>sidereal</u> orbital period of the Moon is 27.32 days.

So we have the formula for acceleration in a circle that needs the distance and velocity, but we now know both of those, so we can say that the Moon's acceleration is this:

$$a = \left(\frac{2\pi}{P}\right)^2 M$$

$$\begin{aligned} a &= v^2/M \\ &= [(2\pi M)/P]^2/M \\ &= (2\pi)^2 M/P^2 \\ &= (2\pi/P)^2 M \end{aligned}$$

Is <u>that</u> of any use?

Well, we know that acceleration due to gravity is what holds the Moon in orbit, so if only we knew how hard the Earth is pulling the Moon, then we would know that.

But we do.

We know that acceleration at the Earth's surface, at distance ***R*** from the centre, is ***g.*** We also know that it falls off as an inverse square. Hence the acceleration due to gravity at any distance, say ***M,*** is given by:

$$a = g\left(\frac{R}{M}\right)^2$$

where R is the Earth's radius.

The "inverse square" bit means this. As you get further from something, the amount of gravitational force it exerts on you is less. Newton's law tells us exactly how much less.

A square that's three times the side-length has nine times the area. In the same way, if you go three times as far from something, the force it exerts on you will be nine times less.

So putting it all together we get:

acceleration in a circle : $a = \left(\frac{2\pi}{P}\right)^2 M$

acceleration due to gravity : $a = g\left(\frac{R}{M}\right)^2$

Therefore $M^3 = g\left(\frac{RP}{2\pi}\right)^2$

$$\left(\frac{2\pi}{P}\right)^2 M = g\left(\frac{R}{M}\right)^2$$

$$M^3 = g\left(\frac{RP}{2\pi}\right)^2$$

And we know everything on the right hand side, except ***g.***

But we can find ***g*** with a pendulum and a stopwatch. (I knew you'd be wondering where they came in.)

We know that the time taken for a complete swing of a pendulum is given by the formula:

$$T = 2\pi\sqrt{L/g}$$

Rearranging this we get:

$$g = L\left(\frac{2\pi}{T}\right)^2$$

We can substitute that into our earlier formula and we get

$$M^3 = L\left(\frac{RP}{T}\right)^2$$

And now we know everything!!

For very small displacements, the force pulling a pendulum back into the vertical is proportional to the amount it's been displaced. More specifically, the ratio of restoring acceleration to gravity is the same as the ratio of displacement to pendulum length. As a formula:

$$x''/g = x/L$$

That means that the formula for its motion (in simple form) can be written as

$$x = a{\cdot}\cos(kt)$$

where ***a*** is the amplitude of the swing, and $k = \sqrt{g/L}$

One cycle is then complete when $kt = 2\pi$ and so one complete cycle of the pendulum takes $2\pi\sqrt{L/g}$ seconds.

Of course we have to go away and construct a pendulum, and then we have to measure how long it takes to swing. Typically we measure 10 swings, both back and forth, and then divide the total time by 10. We should also do that several times to make sure we get error bars on the result, because each one will vary slightly. There's lots to do here.

So what do we get? Here are my results:

Length of the pendulum	:	0.45 metres
Period of a pendulum	:	1.345 seconds
Moon's orbital period	:	27.32 x 86400 seconds
Radius of the Earth	:	$40{\times}10^6/(2\pi)$ metres

A moment's work with a calculator, and we compute that the distance to the Moon is 383 thousand kilometres.

Which is the right answer.

Of course, the Moon's orbit isn't circular, the Earth isn't of constant radius, nor is it a sphere, and we've assumed that the metre is one ten millionth of the distance from the North Pole to the Equator. But even so, we're not just in the right ball park, we're smack in the middle of the true range.

Not bad for a few sums.

As a final note to comment on the fundamental inter-connectedness of things in mathematics and science, the correction for sidereal time is related to a problem from Martin Gardner's Mathematical Circus:

If you roll a coin around a fixed coin of the same size, keeping the rims together to prevent sliding, how many rotations will it make in a round trip?